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Filtering the eigenvalues at infinite from the linear stability analysis of incompressible flows

J.V. Valério [a], M.S. Carvalho [a,*], C. Tomei [b]

[a] Department of Mechanical Engineering, PUC-Rio, Rua Marquês de São Vicente 225, Gávea, Rio de Janeiro, RJ 22453-900, Brazil
[b] Department of Mathematics, PUC-Rio, Rua Marquês de São Vicente 225, Gávea, Rio de Janeiro, RJ 22453-900, Brazil

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Abstract

Steady state, two-dimensional flows may become unstable under two and three-dimensional disturbances if the flow parameters exceed some critical values. In many practical situations, determining the parameters at which the flow becomes unstable is essential. Linear hydrodynamic stability of a laminar flow leads to a generalized eigenvalue problem (GEVP) where the eigenvalues correspond to the rate of growth of the disturbances and the eigenfunctions to the amplitude of the perturbation. Solving GEVP's is challenging, because the incompressibility of the liquid gives rise to singularities leading to non-physical *eigenvalues at infinity* that require substantial care. The high computational cost of solving the GEVP has probably discouraged the use of linear stability analysis of incompressible flows as a general engineering tool for design and optimization.

In this work, we propose a new procedure to eliminate the eigenvalues at infinity from the GEVP associated to the linear stability analysis of incompressible flow. The procedure takes advantage of the structure of the matrices involved and avoids part of the computational effort of the standard mapping techniques used to compute the spectrum of incompressible flows. As an example, the method is applied in the solution of linear stability analysis of plane Couette flow.

Keywords: Eigenvalues; Stability analysis; Matrix transformation; Incompressible flows

1. Introduction

Thorough understanding of viscous flows in many situations requires not only the two-dimensional, steady-state solution of the governing equations, but also the sensitivity of those flows to small upsets and to episodic perturbations, i.e. stability analysis. For example, the required stability of a flow that occurs in many manufacturing processes gives rise to bounds on some operation parameters.

In many situations, an asymptotic analysis with respect to infinitesimal disturbances is sufficient to predict the critical flow parameters at which a two-dimensional steady flow becomes unstable. There are many

* Corresponding author. Tel.: +55 21 3527 1174; fax: +55 21 3527 1165.
E-mail address: msc@mec.puc-rio.br (M.S. Carvalho).

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examples of such analysis in the literature. Ruschak [8], Christodoulou and Scriven [4], Coyle et al. [5] and Carvalho and Scriven [3] studied stability analysis of different coating flows. Ramanan and Homsy [7] studied the linear stability of the flow inside a lid-driven cavity, and Severtson and Aidun [10] analyzed the stability of stratified liquid layers in inclined channels. Linear stability analysis proceeds in a familiar fashion. One first considers the linearization of the governing equations about the steady-state flow. The perturbation variables are described by a linear system of coupled differential equations. The discretization of the system of linear differential equations that describe the amplitude of the perturbations and its rate of growth usually leads to a non-Hermitian, generalized eigenvalue problem (GEVP) of the form:

$$\mathbf{Jc} = \sigma\mathbf{Mc}.$$

where the eigenvalue σ is the growth rate of the disturbances. The matrices $\mathbf{J}$ and $\mathbf{M}$ are usually referred to as the *jacobian* and *mass matrices*.

To find the solution of the GEVP is a computationally challenging task. The level of discretization needed to describe the perturbed fields is high, giving rise to large matrices. The large dimension of the problem rules out the calculation of the full spectrum: usually, only the leading eigenvalues (those with the largest real part) are calculated. Iterative methods are used to compute the relevant part of the spectrum. Moreover, the mass matrix $\mathbf{M}$, which is associated with the transient terms of the governing equations, is singular because the continuity equation for incompressible flows does not have a time dependent term. This singularity is responsible for the so-called *eigenvalues at infinity*, which complicate the numerics because most iterative methods favor the eigenvalues with the largest modulus, not those with the largest real part. Ideally, such non-physical eigenvalues at infinity should be eliminated before proceeding to the numerical eigenproblem.

The most effective techniques to solve GEVP are based on some form of preconditioning and Krylov subspace projection methods, such as Arnoldi's and Lanczos methods (see [9]). A simple way to handle the eigenvalues at infinity is to use the shift-and-invert iteration appropriately, so as to map the eigenvalues at infinity to zero. Christodoulou and Scriven [4] used approximately exponential preconditioning by rational transformation for the same purpose. The eigenvalues of the transformed problem are the exponentials of the original eigenvalues, and consequently this transformation maps leading eigenvalues of the original problem to ones of largest modulus, which are favored by the iterative procedures, like Arnoldi's algorithm. In the same spirit, for the study of linear stability of large power systems, Ushida and Nagao [12] use Cayley-type transforms, which convert the eigenvalues at infinity to eigenvalues at zero. All the proposed techniques are computationally expensive and do not eliminate the eigenvalues at infinity from the problem: the dimension of the transformed eigenproblem is the same as the original one. The eigenvalues are only mapped to a part of the spectrum of the transformed eigenproblem that will not be favored by the iterative methods.

In this work, a more careful consideration of how mass and jacobian matrices are constructed indicates the possibility of eliminating the eigenvalues at infinity. The original generalized eigenproblem (GEVP) is converted into a simpler eigenproblem (EVP) whose dimension is smaller than the original one, so that both problems have the same finite spectrum, the smaller one having no eigenvalues at infinity. Unlike the condensation procedure used by Ruschak [8] and Coyle et al. [5] for viscous free surface flows, and by Arora and Sureshkumar [1] for viscoelastic flows, the method proposed in this work is not limited to vanishing Reynolds number. The method does not include a penalty term in the mass conservation equation, as the compressible flow formulation proposed by Sureshkumar [11]. It reduces not only the memory requirement but also the CPU time needed to compute the leading eigenvalues of incompressible viscous flows.

A stream-function formulation is usually employed in parallel flows and the resulting system of equations can be written as a fourth-order differential equation, the so-called Orr-Sommerfeld operator, frequently solved with spectral methods. Here, a more general approach is used. The formulation of the linear equations that describe the perturbation of a steady-state solution, presented in Section 2, is written in terms of the primitive variables $\mathbf{v}$ and p, and then the Galerkin's finite element method was chosen to discretize the equations. The procedure to eliminate the eigenvalues at infinity, based on a two-sided Gaussian elimination, in shown in Section 3. As an example, in Section 4, the method is applied to the solution of the equations related to the linear stability analysis of plane Couette flow. Conclusions are presented in Section 5.

2. Linear stability analysis of viscous flow

2.1. Formulation

The velocity $\mathbf{v}$ and pressure p fields of two-dimensional, steady state, incompressible flow are governed by the continuity and momentum equations:

$$\nabla \cdot \mathbf{v} = 0, \tag{1}$$

$$Re\mathbf{v} \cdot \nabla\mathbf{v} = -\nabla p + \nabla \cdot \boldsymbol{\tau}. \tag{2}$$

The Reynolds Number $Re \equiv \rho VL/\mu$ characterizes the ratio of inertial to viscous forces; V and L are suitable characteristic values of velocity and length, ρ is the liquid density and μ, the liquid viscosity. We denote by $\boldsymbol{\tau} \equiv \nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}$ the viscous stress tensor for Newtonian fluid.

The goal of linear stability analysis is to determine if a two-dimensional, steady flow is stable with respect to infinitesimal disturbances. The disturbed (velocity and pressure) fields are written as the sum of the base state and an infinitesimal perturbation:

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{v_0}(\mathbf{x}) + \epsilon\mathbf{v}'(\mathbf{x})\mathrm{e}^{\sigma t}, \tag{3}$$

$$p(\mathbf{x}, t) = p_0(\mathbf{x}) + \epsilon p'(\mathbf{x})\mathrm{e}^{\sigma t}. \tag{4}$$

Here, $\mathbf{v}_0$ and p_0 are the velocity and pressure fields of the base flow, i.e. the two-dimensional, steady-state solution, which is known *a priori*. The fields $\mathbf{v}'$ and p' describe the amplitudes of the perturbation and σ is the growth factor. If the real part $\Re(\sigma)$ is positive, the disturbance grows with time and the flow is unstable. The velocity $\mathbf{v}$ and pressure p of the disturbed flow are governed by the time-dependent Navier–Stokes system:

$$\nabla \cdot \mathbf{v} = 0, \tag{5}$$

$$Re\left[\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla\mathbf{v}\right] = -\nabla p + \nabla \cdot [\nabla\mathbf{v} + \nabla\mathbf{v}^{\mathrm{T}}], \tag{6}$$

with the appropriate boundary conditions.

A system of linear differential equations for the perturbed fields is obtained after substituting the perturbed fields, e.g. Eqs. (3), and (4), onto the transient Navier–Stokes system and neglecting terms of order $\mathcal{O}(\epsilon^2)$:

$$\nabla \cdot \mathbf{v}' = 0, \tag{7}$$

$$Re[\sigma\mathbf{v}' + \mathbf{v_0} \cdot \nabla\mathbf{v}' + \mathbf{v}' \cdot \nabla\mathbf{v_0}] = -\nabla p' + \nabla \cdot [\nabla\mathbf{v}' + \nabla\mathbf{v}'^{\mathrm{T}}], \tag{8}$$

The unknowns of the problem are the perturbed fields $\mathbf{v}'$ and p' and the growth factor of the perturbation σ.

2.2. Discretization by Galerkin's method and finite element basis functions

The perturbation fields $\mathbf{v}'$, p' and the rate of growth σ may be computed by applying Galerkin's weighted residual method to Eqs. (7) and (8). The weighting functions used for the momentum equation ϕ_j and continuity equations χ_j are piecewise Lagrangian biquadratic and bilinear discontinuous polynomial basis functions, respectively. The weighted residual equations of continuity and each component of the momentum conservation are

$$R_c^j = \int_\Omega \left(\frac{\partial u_h'}{\partial x} + \frac{\partial v_h'}{\partial y}\right)\chi_j \,\mathrm{d}\Omega, \tag{9}$$

$$\begin{aligned} R_{mx}^j = \sigma \int_\Omega Re u_h'\phi_j \,\mathrm{d}\Omega \; + \int_\Omega Re\left[u_0\frac{\partial u_h'}{\partial x} + v_0\frac{\partial u_h'}{\partial y} + u_h'\frac{\partial u_0}{\partial x} + v_h'\frac{\partial u_0}{\partial y}\right]\phi_j + \left[-p_h' + 2\frac{\partial u_h'}{\partial x}\right]\frac{\partial\phi_j}{\partial x} \\ + \left[\frac{\partial u_h'}{\partial y} + \frac{\partial v_h'}{\partial x}\right]\frac{\partial\phi_j}{\partial y}\,\mathrm{d}\Omega - \int_\Gamma [\mathbf{n} \cdot (-p' + \boldsymbol{\tau}')]_x\phi_j \,\mathrm{d}\Gamma, \end{aligned} \tag{10}$$

$$R^j_{my} = \sigma \int_\Omega Re\ v'_h\ \phi_j\ \mathrm{d}\Omega\ + \int_\Omega Re\left[u_0\frac{\partial v'_h}{\partial x} + v_0\frac{\partial v'_h}{\partial y} + u'_h\frac{\partial v_0}{\partial x} + v'_h\frac{\partial v_0}{\partial y}\right]\ \phi_j + \left[-p'_h + 2\frac{\partial v'_h}{\partial y}\right]\frac{\partial \phi_j}{\partial y}$$
$$+ \left[\frac{\partial u'_h}{\partial y} + \frac{\partial v'_h}{\partial x}\right]\frac{\partial \phi_j}{\partial x}\ \mathrm{d}\Omega - \int_\Gamma\ [\mathbf{n}\cdot(-p' + \boldsymbol{\tau}')]_y\phi_j\ \mathrm{d}\Gamma. \tag{11}$$

The flow is defined on a two-dimensional domain Ω, bounded by the curve Γ. Each perturbed field is approximated with a linear combination of the same basis functions:

$$\mathbf{u}'_h = \begin{bmatrix} u'_h \\ v'_h \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^n U_k\phi_k \\ \sum_{k=1}^n V_k\phi_k \end{bmatrix}, \quad p'_h = \sum_{k=1}^m P_k\ \chi_k.$$

Once all the variables are represented in terms of the basis functions, the system of partial differential equations reduces to simultaneous algebraic equations for the coefficients of the basis functions of all fields and the growth rate σ. The number of algebraic equations is $N = 2n + m$, where n is the number of basis functions used to expand each component of the velocity perturbation and m is the number of basis functions used to expand the pressure disturbance. In vector form, the set of algebraic equations is represented by

$$\mathbf{R}(\mathbf{c}) = 0,$$

where $\mathbf{R}$ is the column vector of weighted residual equations and $\mathbf{c}$ is the column vector of coefficients of the finite element basis function with which the perturbation of velocity and pressure are represented:

$$\mathbf{R} = [R^1_{mx}, R^2_{mx}, \ldots, R^n_{mx}, R^1_{my}, R^2_{my}, \ldots, R^n_{my}, R^1_c, R^2_c, \ldots, R^m_c]^{\mathrm{T}},$$
$$\mathbf{c} = [U_1, U_2, \ldots, U_n, V_1, V_2, \ldots, V_n, P_1, P_2, \ldots, P_m]^{\mathrm{T}}.$$

Here, the algebraic equations resulting from the discretization process are ordered so that the first n equations are associated with the residuals from the horizontal component of the conservation of momentum, the next n, with residuals from the vertical component, and the last m, with the residuals from the conservation of mass. In the same way, the coefficients from the finite element expansion – the unknowns of the problem – are ordered so that the n coefficients from the horizontal velocity component appear first, followed by the n coefficients form the vertical component, at the end, the m coefficients from the pressure expansion.

When this set of equations is expanded in Taylor series and truncated at order $\mathcal{O}(\mathbf{c}^2)$ on the grounds that the perturbation is infinitesimal, one obtains

$$\frac{\partial \mathbf{R}}{\partial \mathbf{c}}\mathbf{c} = 0.$$

Notice that $\frac{\partial \mathbf{R}}{\partial \mathbf{c}}$ is the matrix of sensitivities of the weighted residuals with respect to the unknown coefficient of the perturbations. For convenience, one writes

$$\frac{\partial \mathbf{R}}{\partial \mathbf{c}} = -\sigma\mathbf{M} + \mathbf{J},$$

where $\mathbf{M}$, which multiplies the growth rate σ, is called the mass matrix and $\mathbf{J}$ is the jacobian matrix. Thus, this discretization of the differential equations of the perturbation fields give rise to the generalized, non-Hermitian eigenproblem

$$\mathbf{J}\mathbf{c} = \sigma\mathbf{M}\mathbf{c}. \tag{12}$$

The block structure of the mass and jacobian matrices can be made explicit by splitting the vector of residual equations, Eqs. (9)–(11), into a transient contribution $\mathbf{Rt}$ and a steady-state contribution $\mathbf{Rs}$, i.e. $\mathbf{R} \equiv \sigma\mathbf{Rt} + \mathbf{Rs}$:

$$
\mathbf{M} = - \begin{array}{cc}
\left(\begin{array}{c|c|c}
\dfrac{\partial Rt^j_{mx}}{\partial U_k} & \dfrac{\partial Rt^j_{mx}}{\partial V_k} = 0 & \dfrac{\partial Rt^j_{mx}}{\partial P_k} = 0 \\
\hline
\dfrac{\partial Rt^j_{my}}{\partial U_k} = 0 & \dfrac{\partial Rt^j_{my}}{\partial V_k} & \dfrac{\partial Rt^j_{my}}{\partial P_k} = 0 \\
\hline
\dfrac{\partial Rt^j_{c}}{\partial U_k} = 0 & \dfrac{\partial Rt^j_{c}}{\partial V_k} = 0 & \dfrac{\partial Rt^j_{c}}{\partial P_k} = 0
\end{array}\right) & \begin{array}{c} n \\ \\ n \\ \\ m \end{array} \\
\begin{array}{ccc} n & \qquad n & \qquad m \end{array} &
\end{array}
$$

$$
\mathbf{J} = \begin{array}{cc}
\left(\begin{array}{c|c|c}
\dfrac{\partial Rs^j_{mx}}{\partial U_k} & \dfrac{\partial Rs^j_{mx}}{\partial V_k} & \dfrac{\partial Rs^j_{mx}}{\partial P_k} \\
\hline
\dfrac{\partial Rs^j_{my}}{\partial U_k} & \dfrac{\partial Rs^j_{my}}{\partial V_k} & \dfrac{\partial Rs^j_{my}}{\partial P_k} \\
\hline
\dfrac{\partial Rs^j_{c}}{\partial U_k} & \dfrac{\partial Rs^j_{c}}{\partial V_k} & \dfrac{\partial Rs^j_{c}}{\partial P_k} = 0
\end{array}\right) & \begin{array}{c} n \\ \\ n \\ \\ m \end{array} \\
\begin{array}{ccc} n & \qquad n & \qquad m \end{array} &
\end{array}
$$

3. Elimination of the eigenvalues at infinity

The linear stability analysis then leads to a generalized eigenproblem (12). The mass matrix $\mathbf{M}$ is block diagonal and singular because the continuity equation for incompressible liquids does not have a time derivative term. Thus, the number of (finite) eigenvalues of (12) is smaller than the dimension of the problem $N = 2n + m$. The missing eigenvalues are commonly referred to as *eigenvalues at infinity*, because if the mass matrix is slightly perturbed to remove the singularity, e.g. $\mathbf{M}^* = \mathbf{M} + \epsilon\mathbf{I}$, large eigenvalues appear in the spectrum, and they grow unbounded as $\epsilon \to 0$. Truncation errors in the numerical methods used to calculate the spectrum of (12) may be interpreted as perturbations of the mass matrix and lead to the appearance of very large eigenvalues, corresponding to the eigenvalues at infinity of the original problem. According to Christodoulou [4], the number of eigenvalues at infinity is equal to the number of algebraic constraints (equations with no time derivative) in the discrete eigenproblem, i.e. the number of rows identically equal to zero in the mass matrix. In viscous flows of incompressible liquids, they would then count the number of continuity residuals (number of degrees of freedom associated with the pressure field) plus the number of essential boundary conditions on the velocity field. As we shall see from our algorithm, there are a few more, being equal to twice the number of residual equations associated with the mass conservation equations (twice the number of degrees of freedom associated with the pressure field) plus the number of residuals associated with the essential boundary conditions on velocity.

3.1. The algorithm

Following the ordering scheme explained before, both the mass and jacobian matrices are divided into blocks. The values m and n indicate the dimension of each block: for example, $[\mathbf{M_{11}}]$ is $n \times n$:

$$\mathbf{M} = \begin{array}{c} \left(\begin{array}{c|c|c} \mathbf{M_{11}} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{M_{22}} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right) \begin{array}{c} n \\ n \\ m \end{array} \\ \begin{array}{ccc} n & n & m \end{array} \end{array} \quad \mathbf{J} = \begin{array}{c} \left(\begin{array}{c|c|c} \mathbf{J_{11}} & \mathbf{J_{12}} & \mathbf{J_{13}} \\ \hline \mathbf{J_{21}} & \mathbf{J_{22}} & \mathbf{J_{23}} \\ \hline \mathbf{J_{31}} & \mathbf{J_{32}} & \mathbf{0} \end{array}\right) \begin{array}{c} n \\ n, \\ m \end{array} \\ \begin{array}{ccc} n & n & m \end{array} \end{array} \tag{13}$$

The eigenvalues σ of the GEVP (12) are the roots of the determinant $p(\sigma)$ of the matrix $\mathbf{A} = \mathbf{J} - \boldsymbol{\sigma}\mathbf{M}$, $p(\sigma) = \det(\mathbf{A})$. Said differently, we are interested in the values of σ for which the homogeneous system $(\mathbf{J} - \sigma\mathbf{M})\mathbf{c} = 0$ has a non-trivial solution $\mathbf{c}$. In particular, if one replaces both $\mathbf{J}$ and $\mathbf{M}$ by matrices $\widetilde{\mathbf{J}}$ and $\widetilde{\mathbf{M}}$, the GEVP $\widetilde{\mathbf{J}}\mathbf{d} = \sigma\widetilde{\mathbf{M}}\mathbf{d}$ has the same (generalized) eigenvalues σ as the original system if the corresponding homogeneous system $(\widetilde{\mathbf{J}} - \sigma\widetilde{\mathbf{M}})\mathbf{d} = 0$ has a non-trivial solution $\mathbf{d}$. Convenient modifications are related to the process of solving the homogeneous system above by a two-sided Gaussian elimination, in the sense that row and column operations are allowed. More algebraically, we consider left and right multiplications of both $\mathbf{J}$ and $\mathbf{M}$ by invertible matrices $\mathbf{X}$ and $\mathbf{Y}$ independent of σ. Notice that we are not interested in the value of the solutions $\mathbf{c}$ or $\mathbf{d}$, but just in the existence of non-trivial ones.

As mentioned before, the b algebraic equations associated with the Dirichlet boundary conditions do not have a time derivative, and the perturbed velocity field at these boundaries are identically zero. Such equations belong to the first two rows of blocks of $\mathbf{J}$ and $\mathbf{M}$. The rows associated with these equations only have a non-zero position, equal to 1, at diagonal entries of matrix $\mathbf{J}$. Calling $\mathbf{J^b}$, $\mathbf{M^b}$ the matrices after elimination of the rows and columns related to these equations and unknowns, the dimension of $\mathbf{A^b} = \mathbf{J^b} - \boldsymbol{\sigma}\mathbf{M^b}$ is $2n + m - b$. The determinants $p^b(\sigma)$ and $p(\sigma)$ are the same. It is convenient to redefine the block structure of $\mathbf{A^b}$ as below:

$$\mathbf{A^b} = \begin{array}{c} \left(\begin{array}{c|c|c} \mathbf{A^b_{11}}(\sigma) & \mathbf{A^b_{12}}(\sigma) & \mathbf{A^b_{13}} \\ \hline \mathbf{A^b_{21}}(\sigma) & \mathbf{A^b_{22}}(\sigma) & \mathbf{A^b_{23}} \\ \hline \mathbf{A^b_{31}} & \mathbf{A^b_{32}} & \mathbf{0} \end{array}\right) \begin{array}{c} m \\ 2n-m-b \\ m \end{array} \\ \begin{array}{ccc} m & 2n-m-b & m \end{array} \end{array} \tag{14}$$

The jacobian matrices $\mathbf{J}$ as well as $\mathbf{J^b}$ are invertible in most situations. The exception is at turning points on the solution path constructed as Reynolds number rises, i.e., for a given flow there are a isolated values of Reynolds number at which the Jacobian matrix may become singular.

The blocks $(\mathbf{A^b_{13}}, \mathbf{A^b_{23}})^{\mathrm{T}}$ and $(\mathbf{A^b_{31}}, \mathbf{A^b_{32}})$ are related, respectively, to the sensitivity of the momentum equations to pressure and to the sensitivity of the continuity equation to velocity. They are independent of the growth factor σ (no contribution from the mass matrix) and Reynolds number Re. By construction, the jacobian matrices $\mathbf{J}$ and $\mathbf{J^b}$ are simultaneously invertible and thus the last m rows and m columns of the matrix $\mathbf{A^b}$ are linearly independent. Because the blocks $(\mathbf{A^b_{13}}, \mathbf{A^b_{23}})^{\mathrm{T}}$ and $(\mathbf{A^b_{31}}, \mathbf{A^b_{32}})$ do not depend on the Reynolds number, their columns and rows are always linearly independent, irrespective of Re. Indeed, if this were not true, the Jacobian matrix would have been singular for all values of Re.

Since the blocks $(\mathbf{A^b_{13}}, \mathbf{A^b_{23}})^{\mathrm{T}}$ and $(\mathbf{A^b_{31}}, \mathbf{A^b_{32}})$ are linearly independent, the sub-matrices $\mathbf{A^b_{13}}$ and $\mathbf{A^b_{31}}$ are invertible or there must be column and row permutations that make them non-singular. Let $\mathbf{A^b_{perm}}$ be the matrix obtained from permuting $\mathbf{A^b}$ in such a way. These blocks now may be used by a two-sided Gaussian elimination to set to zero the blocks in positions $\mathbf{A^b_{23}}$ and $\mathbf{A^b_{32}}$. More precisely, define

$$\widetilde{\mathbf{A}} = \left(\begin{array}{c|c|c} \mathbf{A^b_{11}}(\sigma) & \widetilde{\mathbf{A}}_{\mathbf{12}}(\sigma) & \mathbf{A^b_{13}} \\ \hline \widetilde{\mathbf{A}}_{\mathbf{21}}(\sigma) & \widetilde{\mathbf{A}}_{\mathbf{22}}(\sigma) & \mathbf{0} \\ \hline \mathbf{A^b_{31}} & \mathbf{0} & \mathbf{0} \end{array}\right) = \mathbf{T}_\ell \, \mathbf{A^b_{perm}} \, \mathbf{T_r}, \tag{15}$$

where

$$\mathbf{T}_\ell = \left(\begin{array}{c|c|c} \mathbf{I}_{[m]} & \mathbf{0} & \mathbf{0} \\ \hline -\mathbf{A^b_{23}}\mathbf{A^b_{13}}^{-1} & \mathbf{I}_{[2n-m-b]} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{I}_{[m]} \end{array}\right), \quad \mathbf{T_r} = \left(\begin{array}{c|c|c} \mathbf{I}_{[m]} & -\mathbf{A^b_{31}}^{-1}\mathbf{A^b_{32}} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I}_{[2n-m-b]} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{I}_{[m]} \end{array}\right). \tag{16}$$

Since the matrices $\mathbf{T}_\ell$ and $\mathbf{T_r}$ are triangular with diagonal entries equal to one, their determinants are equal to one. The polynomial $p_1(\sigma)$ of the transformed matrix $\widetilde{\mathbf{A}}$ is equal to the original polynomial, up to an irrelevant sign:

$$p_1(\sigma) = \det(\widetilde{\mathbf{A}}) = \det(\mathbf{T}_\ell)\ \det(\mathbf{A^b_{perm}})\det(\mathbf{T_r}) = \det(\mathbf{A^b_{perm}}) = \pm p(\sigma). \tag{17}$$

Again, the multiplication of $\mathbf{A^b_{perm}}$ by $\mathbf{T}_\ell$ and $\mathbf{T_r}$ does not change the spectrum of the original problem. Also, the determinant of the transformed matrix $\widetilde{\mathbf{A}}$ may be computed by permuting rows, so as to interchange the bottom and top row blocks. Indeed, this gives rise to a block triangular matrix, whose determinant, up to sign, equals $\det(\widetilde{\mathbf{A}}_{\mathbf{13}})\det(\widetilde{\mathbf{A}}_{\mathbf{31}})\det(\widetilde{\mathbf{A}}_{\mathbf{22}}(\sigma))$. The upshot is that $p(\sigma)$, the original polynomial, and $\det(\widetilde{\mathbf{A}}_{\mathbf{22}}(\sigma))$ have the same roots.

Generically, the matrix $\widetilde{\mathbf{A}}_{\mathbf{22}} = -\sigma\widetilde{\mathbf{M}}_{\mathbf{22}} + \widetilde{\mathbf{J}}_{\mathbf{22}}$ is non-singular, in the sense that $\widetilde{\mathbf{M}}_{\mathbf{22}}$ is invertible. Thus the number of roots of its characteristic polynomial is $2n - m - b$, the number of finite eigenvalues of the original problem. Consequently, the number of eigenvalues at infinity of the original problem is twice the number of degrees of freedom associated with the mass conservation equation plus the eigenvalues at infinity that come from essential boundary conditions, i.e. $2m + b$.

Now, the finite portion of the spectrum of the GEVP (12) can be calculated by solving the smaller GEVP:

$$(-\sigma\ \widetilde{\mathbf{M}}_{22} + \widetilde{\mathbf{J}}_{22})\mathbf{c_2} = 0, \tag{18}$$

where the $(2n - m - b) \times (2n - m - b)$ matrices $\widetilde{\mathbf{M}}_{22}$ and $\widetilde{\mathbf{J}}_{22}$ are given by

$$\widetilde{\mathbf{J}}_{22} = (-\mathbf{J^b_{23}}\mathbf{J^b_{13}}^{-1}\mathbf{J^b_{11}} + \mathbf{J^b_{21}})(-\mathbf{J^b_{31}}^{-1}\mathbf{J^b_{32}}) + (-\mathbf{J^b_{23}}\mathbf{J^b_{13}}^{-1})\mathbf{J^b_{12}} + \mathbf{J^b_{22}}; \tag{19}$$

$$\widetilde{\mathbf{M}}_{22} = (-\mathbf{J^b_{23}}\mathbf{J^b_{13}}^{-1}\mathbf{M^b_{11}} + \mathbf{M^b_{21}})(-\mathbf{J^b_{31}}^{-1}\mathbf{J^b_{32}}) + (-\mathbf{J^b_{23}}\mathbf{J^b_{13}}^{-1})\mathbf{M^b_{12}} + \mathbf{M^b_{22}}. \tag{20}$$

Since $\widetilde{\mathbf{M}}_{\mathbf{22}}$ is invertible, there is an equivalent simple eigenvalue problem (EVP),

$$\underbrace{\widetilde{\mathbf{M}}_{22}^{-1}\widetilde{\mathbf{J}}_{22}}_{\mathbf{D}}\mathbf{c_2} = \sigma\mathbf{c_2}. \tag{21}$$

As stated, the EVP requires the inversion of two $m \times m$ matrices and one $(2n - m - b) \times (2n - m - b)$ matrix. Clearly, for many iterative procedures related to the computation of eigenvalues, inversions may be replaced by solving linear systems which special structure.

The generalized eigenvectors $\mathbf{c}$ of the original problem $(\mathbf{J} - \sigma\mathbf{M})\mathbf{c} = 0$ are computed as a function of the eigenvectors $\mathbf{c}_2$ of the transformed smaller problem $(-\sigma\ \widetilde{\mathbf{M}}_{22} + \widetilde{\mathbf{J}}_{22})\mathbf{c_2} = 0$, as shown next.

The first step of the proposed method was to eliminate the rows and columns associated with the Dirichlet boundary conditions. The corresponding eigenvector $\mathbf{c^b}$ of the problem $\mathbf{A^b}\mathbf{c^b} = 0$ is simply the original eigenvector without the zero entries associated with the Dirichlet boundary conditions. The eigenvector $\mathbf{c^b_{perm}}$ of the permuted matrix $\mathbf{A^b_{perm}}$ differs from $\mathbf{c^b}$ by the column permutation performed in matrix $\mathbf{A^b}$.

Since $\widetilde{\mathbf{A}} = \mathbf{T}_\ell \mathbf{A}^{\mathbf{b}}_{\mathbf{perm}} \mathbf{T}_{\mathbf{r}}$ with $\det(\mathbf{T}_\ell) = 1$ and $\det(\mathbf{T}_{\mathbf{r}}) = 1$, the eigenproblem can be re-written as

$$\mathbf{T}_\ell \mathbf{A}^{\mathbf{b}}_{\mathbf{perm}} \underbrace{\mathbf{T}_{\mathbf{r}} \mathbf{d}}_{\mathbf{c}^{\mathbf{b}}_{\mathbf{perm}}} = 0 \Rightarrow \widetilde{\mathbf{A}}\mathbf{d} = 0, \tag{22}$$

where $\mathbf{d} \equiv \mathbf{T}_{\mathbf{r}}^{-1} \mathbf{c}^{\mathbf{b}}_{\mathbf{perm}}$ is the generalized eigenvector of $\widetilde{\mathbf{A}}$. It is convenient to split it respecting the block structure of $\widetilde{\mathbf{A}}$:

$$\mathbf{d} = \begin{array}{c} \begin{pmatrix} \mathbf{d_1} \\ \mathbf{d_2} \\ \mathbf{d_3} \end{pmatrix} \end{array} \begin{array}{l} m \\ 2n \quad -m \quad -b \\ m \end{array}$$

The system (22) may be written in terms of the blocks of $\widetilde{\mathbf{A}}$ and $\mathbf{d}$:

$$\begin{cases} \mathbf{A}^{\mathbf{b}}_{\mathbf{11}} \mathbf{d_1} + \widetilde{\mathbf{A}}_{\mathbf{12}} \mathbf{d_2} + \mathbf{A}^{\mathbf{b}}_{\mathbf{13}} \mathbf{d_3} & = 0, \\ \widetilde{\mathbf{A}}_{\mathbf{21}} \mathbf{d_1} + \widetilde{\mathbf{A}}_{\mathbf{22}} \mathbf{d_2} & = 0, \\ \mathbf{A}^{\mathbf{b}}_{\mathbf{31}} \mathbf{d_1} & = 0. \end{cases}$$

The matrix $\mathbf{A}^{\mathbf{b}}_{\mathbf{31}}$ is invertible so the only solution for $\mathbf{A}^{\mathbf{b}}_{\mathbf{31}}\ \mathbf{d_1} = 0$ is the trivial solution $\mathbf{d_1} = 0$. The second equation becomes $\widetilde{\mathbf{A}}_{\mathbf{22}} \mathbf{d_2} = 0$. Notice that this equation corresponds to the reduced generalized eigenproblem (18). The non-trivial solution is the eigenvector $\mathbf{c_2}$. The block $\mathbf{d_3}$ can be easily obtained: $\mathbf{d_3} = -\mathbf{A}^{\mathbf{b}^{-1}}_{\mathbf{13}} \widetilde{\mathbf{A}}_{\mathbf{12}} \mathbf{c_2}$.

Thus, the eigenvector of the large eigenproblem after the left and right matrix multiplication $\mathbf{d}$ is written in terms of the eigenvector of the reduced eigenproblem $\mathbf{c_2}$ as

$$\mathbf{d} = \begin{pmatrix} \mathbf{0} \\ \mathbf{c_2} \\ \mathbf{d_3} = -\mathbf{A}^{\mathbf{b}^{-1}}_{\mathbf{13}} \widetilde{\mathbf{A}}_{\mathbf{12}} \mathbf{c_2} \end{pmatrix}. \tag{23}$$

From this equation, one computes the generalized eigenvectors associated with the previous modifications of the original problem, obtaining $\mathbf{c}$ without difficulty.

4. An example: stability of plane Couette flow

4.1. Perturbed equations and solution method

The method described in the previous section is applied to study the stability of plane Couette flow. The flow geometry and boundary conditions are shown in Fig. 1: liquid flows between two parallel plates located at $y = \pm 1$ that are moving with velocity $U = \pm 1$. The steady-state solution is

$$\mathbf{v_0} = (y, 0, 0) \quad \text{and} \quad p_0 = 0. \tag{24}$$

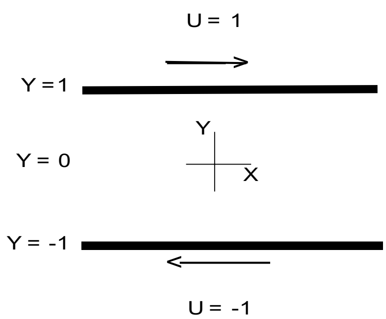

Fig. 1. Configuration of plane Couette flow.

The base flow is perturbed as

$$\mathbf{v}(x,y,t) = \mathbf{v_0}(y) + \epsilon\, \mathbf{v}'(y)\mathrm{e}^{\mathrm{i}\alpha x+\sigma t} \quad \text{and} \quad p(x,y,t) = p_0(y) + \epsilon\, p'(y)\mathrm{e}^{\mathrm{i}\alpha x+\sigma t}. \tag{25}$$

For simplicity, only two-dimensional perturbations were considered. Let α be the wavelength of the periodic perturbation along the flow direction. Substituting the perturbed fields into the transient conservations equations and neglecting the higher order terms ($\mathcal{O}(\epsilon^2)$), a system of differential equations on the perturbed variables, e.g. $\mathbf{v}' = (u',v')$ and p', is obtained:

$$\mathrm{i}\alpha u' + \frac{\mathrm{d}v'}{\mathrm{d}y} = 0, \tag{26}$$

$$Re[(\sigma + \mathrm{i}\alpha y)u' + v'] = -\mathrm{i}\alpha p' + \frac{\mathrm{d}^2 u'}{\mathrm{d}y^2} - \alpha^2 u', \tag{27}$$

$$Re[(\sigma + \mathrm{i}\alpha y)v'] = -\frac{\mathrm{d}p'}{\mathrm{d}y} + \frac{\mathrm{d}^2 v'}{\mathrm{d}y^2} - \alpha^2 v'. \tag{28}$$

The pressure field may be eliminated from the equations, yielding a fourth-order Orr-Sommerfeld operator for the amplitude of the perturbed stream function.

It is common practice to analyze the stability of Couette flow by discretizing the Orr-Sommerfeld operator using spectral methods. An example of such procedure is presented by Dongarra et al. [6], that used Chebyshev-τ method to discretize the Orr-Sommerfeld equation and the QZ method to solve the generalized eigenvalue problem. The authors acknowledged that the singularity of the mass matrix $\mathbf{M}$ might account for the appearance of eigenvalues at infinity.

Here, instead, the stability analysis is formulated in terms of the primitive variables, i.e. velocity and pressure, as in Eqs. (26)–(28). At a fixed wave number α, the amplitudes of the perturbations u', v' and p', and their growth rate σ are found by applying Galerkin's weighted residual method to Eqs. (26)–(28), as explained in the previous section. The velocity perturbations are expanded using piecewise Lagrangian quadratic polynomials ϕ_j and the pressure perturbation are approximated using piecewise linear discontinuous polynomials χ_j.

The resulting mass and jacobian matrices have the same structure presented in (13). The entries of each block are given by

$$\mathbf{M_{11}} = -\frac{\partial Rt^j_{mx}}{\partial U_k} = -\int_{-1}^{1} Re\phi_j\phi_k \mathrm{d}y;$$

$$\mathbf{M_{22}} = -\frac{\partial Rt^j_{my}}{\partial V_k} = -\int_{-1}^{1} Re\phi_j\phi_k \mathrm{d}y;$$

$$\mathbf{J_{11}} = \frac{\partial Rs^j_{mx}}{\partial U_k} = \int_{-1}^{1} \left((Re\mathrm{i}\alpha y + \alpha^2)\phi_j\phi_k + \frac{\mathrm{d}\phi_j}{\mathrm{d}y}\frac{\mathrm{d}\phi_k}{\mathrm{d}y} \right) \mathrm{d}y;$$

$$\mathbf{J_{12}} = \frac{\partial Rs^j_{mx}}{\partial V_k} = \int_{-1}^{1} Re\phi_j\phi_k \,\mathrm{d}y;$$

$$\mathbf{J_{13}} = \frac{\partial Rs^j_{mx}}{\partial P_k} = \int_{-1}^{1} \mathrm{i}\alpha\chi_j\phi_k \,\mathrm{d}y;$$

$$\mathbf{J_{22}} = \frac{\partial Rs^j_{my}}{\partial V_k} = \int_{-1}^{1} \left((Re\mathrm{i}\alpha y + \alpha^2)\phi_j\phi_k + \frac{\mathrm{d}\phi_j}{\mathrm{d}y}\frac{\mathrm{d}\phi_k}{\mathrm{d}y} \right) \mathrm{d}y;$$

$$\mathbf{J_{23}} = \frac{\partial Rs^j_{my}}{\partial P_k} = -\int_{-1}^{1} \chi_j \frac{\mathrm{d}\phi_k}{\mathrm{d}y} \mathrm{d}y;$$

$$\mathbf{J_{31}} = \frac{\partial Rs^j_{c}}{\partial U_k} = \int_{-1}^{1} i\alpha\phi_j\chi_k \,\mathrm{d}y;$$

$$\mathbf{J_{32}} = \frac{\partial Rs^j_{c}}{\partial V_k} = \int_{-1}^{1} \frac{\mathrm{d}\phi_j}{\mathrm{d}y}\chi_k \,\mathrm{d}y;$$

all omitted block entries are identically zero. At each row corresponding to the Dirichlet boundary condition applied at both walls, e.g. $u' = 0$ and $v' = 0$, all the entries of the mass and jacobian matrices are equal to zero, except the diagonal entry of the jacobian matrix, which is equal to one.

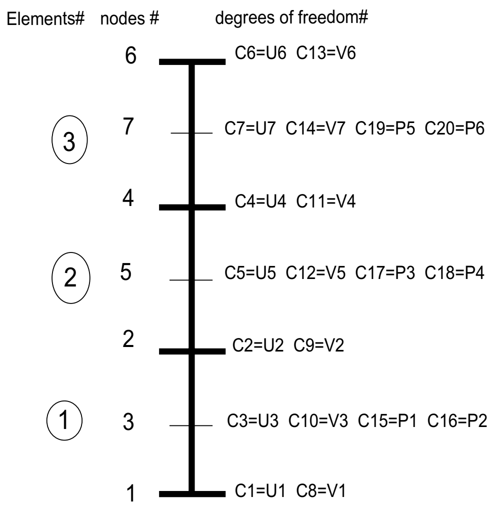

Fig. 2. Numbering scheme for 3 elements, 7 nodes and 20 degrees of freedom: 7 for the x-velocity U, 7 for the y-velocity V and 6 for the pressure P. The 20 related coefficients $C1, \ldots, C20$ are inserted in the matrix as follows: first $u'_h = \sum_{k=1}^{7} U_k \phi_k$, then $v'_h = \sum_{k=1}^{7} V_k \phi_k$ and finally $p'_h = \sum_{k=1}^{6} P_k\ \chi_k$.

4.2. Filtering the eigenvalues at infinity

As an illustration, we consider first only three finite elements. For this discretization level, the number of finite element coefficients (number of unknowns) used to expand each component of the velocity perturbation is $n = 7$, and the number of coefficients to expand the pressure perturbation is $m = 6$. The total number of degrees of freedom of the problem is $N = 2n + m = 20$. The scheme used to number the elements, nodes and degrees of freedom of the problem is illustrated in Fig. 2.

The structure of the non-zero entries of the matrix $\mathbf{A} = -\sigma\mathbf{M} + \mathbf{J}$ is shown in Fig. 3a. The only entries in rows 1, 6, 8 and 13 (the ones associated with the Dirichlet boundary conditions) different than zero are the diagonal elements, that are equal to one. As explained before, the first step is to remove the rows and columns associated with the essential boundary conditions. The structure of the resulting matrix, $\mathbf{A}^{\mathbf{b}}$ is shown in Fig. 3b. The next step is to eliminate the blocks $\mathbf{A}^{\mathbf{b}}_{\mathbf{32}}$ and $\mathbf{A}^{\mathbf{b}}_{\mathbf{23}}$ using the transformation defined in Eq. (16). In order to construct the transformation matrices $\mathbf{T}_\ell$ and $\mathbf{T}_\mathbf{r}$, the inverse of the blocks $\mathbf{A}^{\mathbf{b}}_{\mathbf{13}}$ and $\mathbf{A}^{\mathbf{b}}_{\mathbf{31}}$ need to be evaluated. Two different approaches may be used at this step. The first is to simply evaluate the inverses of each block. The alternative approach is to perform column and row permutations in the rectangular matrices $[\mathbf{A}^{\mathbf{b}}_{\mathbf{31}}, \mathbf{A}^{\mathbf{b}}_{\mathbf{32}}]$ and $[\mathbf{A}^{\mathbf{b}}_{\mathbf{13}}, \mathbf{A}^{\mathbf{b}}_{\mathbf{23}}]^{\mathrm{T}}$ in order to reduce the bandwith of each block and minimize the round-off error that comes with the inversion of the matrices.

In a one-dimensional problem like this one, column and row permutations give rise to the matrix structure shown in Fig. 3c: the blocks $\mathbf{A}^{\mathbf{b}}_{\mathbf{perm,13}}$ and $\mathbf{A}^{\mathbf{b}}_{\mathbf{perm,31}}$ become diagonal matrices, with trivial inverses. The structure of the resulting transformed matrix $\widetilde{\mathbf{A}} = \mathbf{T}_\ell\mathbf{A}^{\mathbf{b}}\mathbf{T}_\mathbf{r}$ is shown in Fig. 4. All the (finite) information on the spectrum of the original problem is contained in the $(2n - m - b) \times (2n - m - b)$ central block in the transformed matrix (4×4 in this particular case), indicated in Fig. 4.

4.3. Results

We compare the spectrum of the plane Couette flow predicted by our method with the one presented by [2]. The analysis was performed at $Re = 500$ and $\alpha = 1.5$. In order to verify the independence of the eigenvalues to the number of elements in the discretization, the spectrum of the original generalized eigenproblem $\mathbf{Jc} = \sigma\mathbf{Mc}$ was solved using the QZ method for two different meshes of 100 and 200 elements. The relative deviation $(|\sigma^i_{100} - \sigma^i_{200}|/|\sigma^i_{200}|)$ between the 35 leading eigenvalues (i.e., the finite numbers with largest real part) are shown in Fig. 6. In the range $-6.5 < \Re(\sigma) < 0$, where the real part of these 35 leading eigenvalues are located,

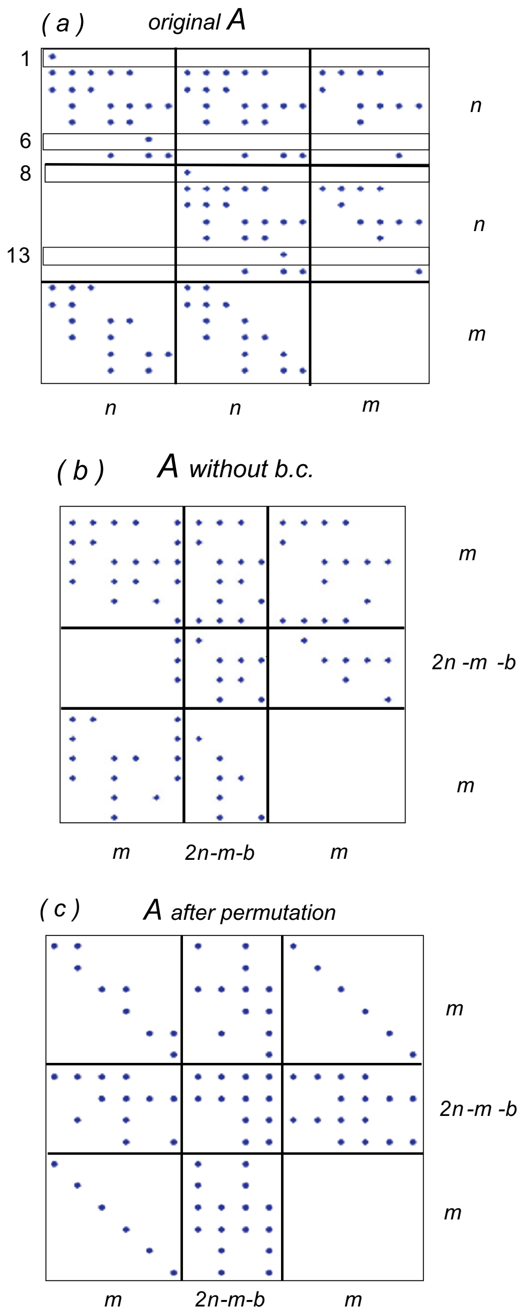

Fig. 3. The structure of the matrix $-\sigma\mathbf{M} + \mathbf{J}$ after each step. (a) the original matrix $\mathbf{A}$; (b) after eliminating rows and columns associated with the Dirichlet boundary conditions $\mathbf{A}$; and (c) after permutation of rows and columns to diagonalize two sub-blocks.

the maximal deviation is less than $\mathcal{O}(10^{-3})$. Thus, a mesh of 100 elements was considered to be fine enough to predict the leading eigenvalues of the problem. Moreover, the deviation decrease for eigenvalues closer to the imaginary axis, reaching $\mathcal{O}(10^{-7})$.

The eigenvalues found by our method are plotted together with Bottaro's spectrum in Fig. 5. A more precise comparison, with several significant digits, is not possible since Bottaro's results were obtained directly from a graph in his work.

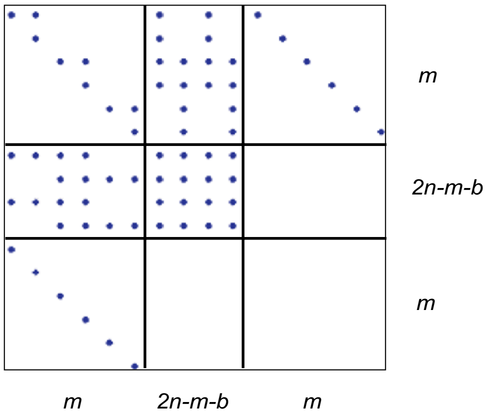

Fig. 4. Structure of the final transformed matrix $\widetilde{\mathbf{A}}$. The (finite) spectrum of the original problem is equals the spectrum of the $(2n-m-b)\times(2n-m-b)$ block.

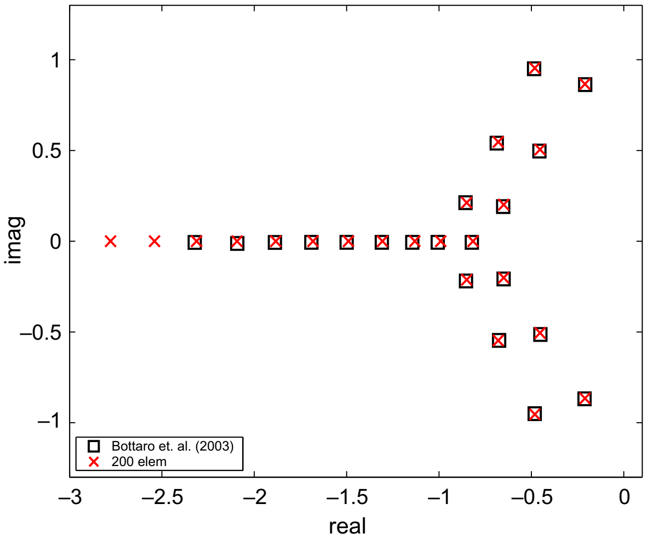

Fig. 5. Leading eigenvalues of plane Couette flow at $Re = 500$ and $\alpha = 1.5$, as reported by Bottaro and computed using the QZ method on the original GEVP with 200 elements.

With a mesh of 100 elements, the number of degrees of freedom of the problem (dimension of the original generalized eigenproblem) is $N = 602$, with $n = 201$, $m = 200$ and $b = 4$. After using the transformations which eliminate the eigenvalues at infinity of the problem, the reduced matrix $\tilde{\mathbf{A}}_{22}$ has dimension $2n - m - b = 198$. Typically, $\mathcal{O}(n) \simeq \mathcal{O}(m)$, so the dimension of the original GEVP is $N = 2n + m \simeq \mathcal{O}(3n)$. The dimension of the reduced EVP is $2n - m - b \simeq \mathcal{O}(n)$. Since the number of essential boundary conditions is much smaller than the number of degrees of freedom associated with each velocity component, i.e. $b << n$, the reduced EVP is approximately 1/3 of the size of the original problem. Notice that there was no effort to try to make use of the intrinsic symmetric in the flow geometry, which might reduce the order of the problems further.

The reduced EVP, Eq. (21), and also the GEVP, Eq. (12), were solved by the LAPACK routine ZGEEV (for non-Hermitian matrices) and, in the case of the GEVP, using QZ method. No special features of the matrices were employed, so as not do distort comparisons between methods. As expected, the finite eigenvalues of the EVP and of the GEVP coincide: their relative deviation is of the order of $\mathcal{O}(10^{-7})$. The modulus of the entries of the eigenvector related to the critical eigenvalue $\sigma_l = -0.20934 + i0.86613$ are shown in Fig. 7. Indeed, the

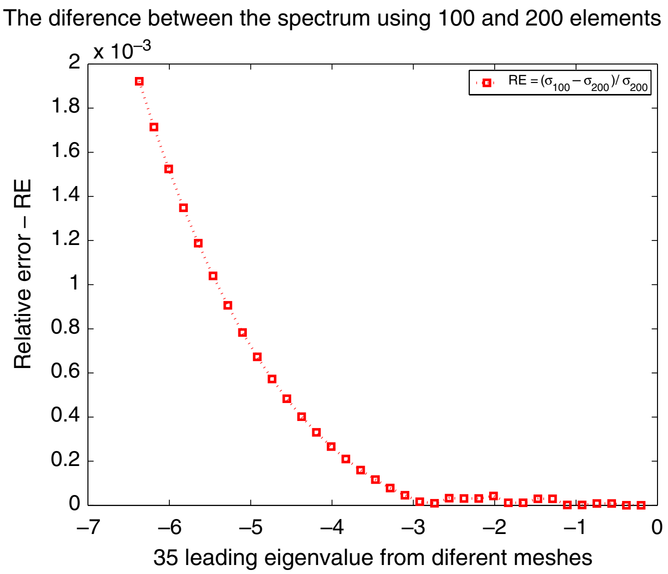

Fig. 6. Relative deviation between the 35 leading eigenvalues for different meshes: it is less than 2×10^{-3} and decreases sharply as the real part goes to zero.

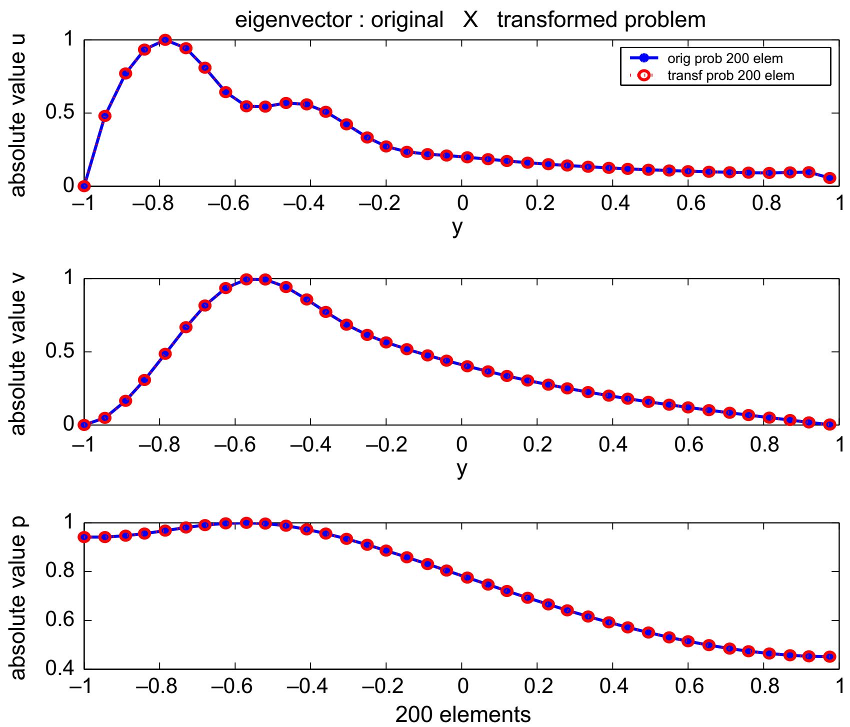

Fig. 7. Eigenvectors of the original GEVP and of the reduced EVP; 200 elements.

method used to eliminate the eigenvalues at infinity was able to accurately predict the eigenpair of the problem essentially to 7 digits.

Table 1
CPU time, in seconds, to compute the eigenvalues: (a) solving the original GEVP by QZ method and (b) solving the reduced EVP using LAPACK routine

# ele	$N_{\text{full}} = 2n + m$	$N_{\text{transf}} = 2n - m - b$	GEVP time	EVP time	$\frac{t_{\text{GEVP}}}{t_{\text{EVP}}}$
50	302	98	3.57	0.27	13.22
100	602	198	28.70	0.891	32.21
150	902	298	98.25	3.36	29.3
200	1202	398	242.84	7.89	30.78
300	1802	598	836.86	27.67	30.24
350	2102	698	1435.51	40.734	35.27

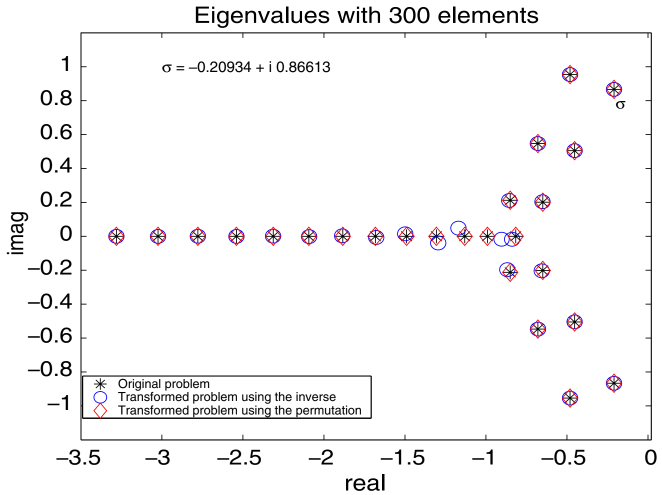

Fig. 8. Leading spectrum computed with 300 elements with and without row and column permutations.

The method reduces significantly the time of computation. Table 1 presents the CPU times, in seconds, required to solve the original GEVP and the reduced EVP, for different meshes. The latter includes the time to compute all the operations necessary to obtain the reduced EVP, which, as mentioned before, consists of inverting two $m \times m$ matrices, one $(2n - m - b) \times (2n - m - b)$ matrix, and some matrix–matrix products. In the one-dimensional problem used as an example in this text, simple row and column permutations transform some off-diagonal sub-blocks into diagonal matrices, trivializing the inverses of the $m \times m$ blocks $\mathbf{A}^{\mathbf{b}}_{13}$ and $\mathbf{A}^{\mathbf{b}}_{31}$. It turns out that the inversion of such blocks before permutations is so simple that the CPU time between both alternatives does not vary substantially. However, the computation of the eigenvalues around $\Re(\sigma) \approx -1$ differ sharply in accuracy for both approaches, as shown in Fig. 8. The proposed method is faster by a factor of approximately 30 for $N_{\text{full}} > 600$. The programming makes use of the sparsity of the matrices. All calculations were performed on a machine with 1.00 GB of RAM and 789 GHz AMD Turion(tm) 64 Mobile processor using MatLab, version 6.5.

5. Final remarks

We present a new method to eliminate the eigenvalues at infinity of the generalized eigenvalue problem that arises from linear stability analysis of incompressible flows. The algorithm transforms the original generalized eigenproblem (GEVP) into an equivalent strict eigenvalue problem (EVP), whose dimension is approximately 1/3 of the original problem. The eigenvalues of the transformed EVP correspond exactly to the finite eigenvalues of the original GEVP. The eigenvectors of the original problem can also be found as a function of the eigenvectors of the reduced problem.

The main advantages of the proposed methods are:

- Eliminates the eigenvalues at infinity without the need of mapping or preconditioning techniques, which are computationally expensive.
- Reduces the size of the eigenproblem without loss of accuracy. Previous methods were restricted to creeping flow analysis (zero Reynolds number) or penalty methods.
- The transformed and smaller mass matrix is non-singular and, consequently, the original GEVP can be easily re-written as an EVP.
- No information about the eigenspaces is lost since the eigenvalues and also the eigenvectors can be found in the reduced problem.

These features bring significant reduction of the computational cost required to evaluate the eigenspectrum of an incompressible flow. In the example presented here, the proposed method was faster by a factor of approximately 30 when compared to the solution of the original GEVP.

The analysis also shows that the number of eigenvalues at infinity of a incompressible viscous flow is actually larger than that proposed by Christodoulou and Scriven [4]; it is equal to twice the number of residual equations associated with the mass conservation equations (twice the number of degrees of freedom associated with the pressure field) plus the number of residuals associated with essential boundary conditions on velocity.

Although the formulation and the example used in this work was based on the linear stability analysis of an incompressible flow, this procedure may be also used to any generalized eigenproblem that comes from linear stability analysis with algebraic restrictions. We intend to expand on these issues in forthcoming publications.

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